B.SC. THIRD SEMESTER (HONS.) EXAMINATION 2021

Subject: Mathematics **Course ID: 32112**

Course Title: Group theory-I **Course Code: SH/MTH/302/C-6**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meanings.

1. **Answer *any five* of the following questions:** **2× 5=10**

a) If $G$is a group of order 22, then prove that G contains an odd number of elements of order 2.

b) Let $\mathbb{R}^-$ denote the set of all negative real numbers. Can you define a binary operation $*$ on $\mathbb{R}^-$ so that $(\mathbb{R}^-,*)$ becomes a group? Justify your answer.

c) Express the following permutation as a product of disjoint cycles in S_8:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 8 & 6 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

d) Prove that $Z(G) = \{a \in G | ag = ga \text{ for all } g \in G\}$ is a subgroup of the group G.

e) Find all the distinct left cosets of $H = 6\mathbb{Z}$ in the group $(\mathbb{Z}, +)$.

f) Let A, B be two normal subgroups of a group G. Then $G/A \simeq G/B$ −prove or disprove.

g) Give an example of an infinite group G such that each of its cyclic subgroups has cardinality maximum 2.

h) Show that $\mathbb{Z}_8$ is not a homomorphic image of $\mathbb{Z}_{15}$.

2. **Answer *any four* of the following questions:** **5× 4=20**

a) Prove that any non-identity permutation $\alpha \in S_n (n \geq 2)$ can be expressed as a product of disjoint cycles, where each cycle is of length ≥ 2.

b) Let S be a non-empty subset of a group G. Let $< S >$ denote the subgroup generated by S in G, i.e., the smallest subgroup containing S in G. Then prove that

$$< S >= \{s_1^{e_1} s_2^{e_2} \dots s_n^{e_n} | \, s_i \in S, e_i = \pm 1, i = 1,2, \dots, n; n \in \mathbb{N}\}.$$

c) Define the normalizer of a subset of a group G. Show that it becomes a subgroup of G. If H is a subgroup of G, then prove that the normalizer of H is the largest subgroup of G in which H is normal. 1+1+3

d) State and prove Cayley's theorem for groups.

e) For any two finite subgroups H, K of a group G, prove that $|HK| = \frac{|H||K|}{|H \cap K|}$.

f) Show that the set of all 8^{th} roots of unity form a cyclic group with respect to the multiplication of complex numbers. Also find all the generators of this group. 3+2

3. Answer *any one* of the following questions: 10× 1=10

a) (i) Prove that any finitely generated subgroup of the group $(\mathbb{Q}, +)$ is cyclic.

(ii) Prove Fermat's little theorem using Lagrange's theorem.

(iii) Let H be a proper subgroup of a group G and $a \in G \setminus H$. Suppose that for all $b \in G$, either $b \in H$ or $Ha = Hb$. Show that H is normal in G.

(iv) Using first isomorphism theorem on a group prove that $\mathbb{R}/\mathbb{Z} \cong S^1$, where S^1 is a unit circle with center $(0,0)$ and radius 1. 3+2+3+2

b) (i) Give an example (with reason) of a non-cyclic, non-commutative group of which each subgroup is cyclic.

(ii) If H is the only subgroup of order m in a group G, then prove that H is normal in G.

(iii) In a group G, if $G/Z(G)$ is cyclic, then show that G is abelian.

(iv) State and prove third isomorphism theorem for groups. 2+2+2+(1+3)

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